CS 6862 Lect 19 April 5, 2011

Continuity Principles

Consider operations $(\mathbb{N} \to \mathbb{N}) \to \mathbb{N}$. We might say

$F(f) = n$ for $f \in (\mathbb{N} \to \mathbb{N})$

If we said $\exists f : \mathbb{N} \to \mathbb{N}. (F(f) = n)$ we might say

$\exists f_0$ a finite function, $f_0 \subseteq f$ and $F(f_0) = n$ to indicate that F uses only a finite amount of its input.

* We could say $f \subseteq f' \supset (F(f) = n \supset F(f') = n)$ to express monotonicity or stability.

The intuitionists do not talk this way about $\mathbb{N} \to \mathbb{N}$ as choice sequences, say α in a spread because α is "growing"

$\alpha(0), \alpha(1), \ldots$

and a functional F uses only some initial segment $\bar{\alpha}(x)$.

They would say F is stable iff * held for finite segments, f, f'.

To say $\exists f\ F(f) = n$ they would say that F is barring in that some $\bar{\alpha}(x)$ is sufficient to produce the output.

Kleene represented $(\mathbb{N} \to \mathbb{N}) \to \mathbb{N}$ in the intuitionistic sense as a mapping $List(\mathbb{N}) \to \mathbb{N}$ and coded $List(\mathbb{N})$ into $\mathbb{N}$.

So functionals were maps $\alpha : \mathbb{N} \to \mathbb{N}$. We say α is continuous, $K_0(\alpha)$, (or $\alpha \in K_0$) iff

stability: (1) $\forall x, y. (\alpha(x) \neq 0 \supset \alpha(x * y) = \alpha(x))$
where $x * y$ appends list x to list y.

and

barring: (2) $\forall \beta\ \exists x (\alpha(\bar{\beta}(x)) \neq 0)$ — to say $\alpha(y) \neq 0$ is to say that α has an integer value since 0 codes nil the empty list.

BC-N $\forall \alpha\ \exists y\ R(\alpha, y) \supset \exists \gamma \in K_0. \forall \alpha. R(\alpha, \gamma(\alpha))$.